

Bulk spectral function sum rule in QCD-like theories with a holographic dual

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We derive the sum rule for the spectral function of the stress-energy tensor in the bulk (uniform dilatation) channel in a general class of strongly coupled field theories. This class includes theories holographically dual to a theory of gravity coupled to a single scalar field, representing the operator of the scale anomaly. In the limit when the operator becomes marginal, the sum rule coincides with that in QCD. Using the holographic model, we verify explicitly the cancellation between large and small frequency contributions to the spectral integral required to satisfy the sum rule in such QCD-like theories.

I. INTRODUCTION

The recent discovery of strongly coupled quark-gluon plasma at RHIC [1–4] has spurred a significant theoretical effort towards understanding the properties of matter described by strongly coupled quantum field theories at finite temperature. The transport properties of such theories are much more sensitive to the strength of the coupling than stationary thermodynamic properties. In particular, the near perfect fluidity in such theories is viewed as a tell-tale sign of the strong coupling.

The transport properties of a theory are closely related to the spectral functions of the operators such as stress-energy tensor. In particular, viscosity can be determined from the low frequency limit of the spectral function by the well-known Kubo formula. The first-principles calculation of the spectral functions using lattice Monte Carlo techniques is a challenging task [5–12], especially in the low-frequency regime, and the prior knowledge of the properties of spectral functions is essential. Therefore, constraints on the spectral functions in the form of the sum rules for the integral of the spectral function has been the subject of recent attention [13–15].

The discovery of the AdS/CFT holographic correspondence [16–18] has opened new possibilities to study thermodynamics and transport in strongly coupled theories (for reviews see [19–25]). Although not yet directly applicable to QCD, at least until we know its holographic dual, these methods allow one to study generic properties of the strongly coupled plasmas using model theories, such as $\mathcal{N} = 4$ SUSY YM, or holographic models which incorporate QCD-like features such as confinement. In particular, the shear channel sum rule derived in [15] has been verified in the $\mathcal{N} = 4$ SUSY YM using AdS/CFT correspondence.

In this paper, we concentrate on the sum rule for the spectral function in the bulk channel, corresponding to uniform dilatation or isotropic expansion. The bulk channel sum-rule is trivial in a conformal theory such as $\mathcal{N} = 4$ SUSY YM. However, nontrivial sum rules in QCD do exist and have been a subject of recent studies [13–15]. Unlike the shear channel, the bulk channel correlation function (and associated bulk viscosity) is sensitive not only to the strength of the coupling, but also to the amount of the scaling violation. Therefore we consider a holographic model with the simplest mechanism of conformality violation, which is similar to QCD. We shall assume that this model describes field theories where scale invariance is broken by the presence of a scalar operator O with dimension $\Delta_+ < 4$ in the action. The scale anomaly, T^μ_μ , is proportional to this operator, and in QCD this role is played by the gluon condensate operator $G_{\mu\nu}G^{\mu\nu}/\alpha_s$.

Thermodynamics and transport in such theories have been first studied in Ref. [26–28]. In particular, it has been shown that the speed of sound $c_s = \sqrt{d\epsilon/dp}$ approaches the conformal value $1/\sqrt{3}$ universally from *below* [29, 30].

The bulk sum rule has been also tested recently in Ref. [31] in a theory whose holographic dual is a dilaton gravity with the Chamblin-Reall dilaton potential [32], which has the virtue of being analytically tractable. However, the mechanism of the scaling violation in such a putative field-theory (if the field-theory dual exists) would differ from that in QCD. In particular, the speed of sound does not approach its conformal value in the high-temperature limit.

In the general class of theories we consider, we shall find that the temperature dependent part of the bulk spectral function $[\rho(\omega)]_T$ obeys the following sum rule

$$\left(3s\frac{\partial}{\partial s} - \Delta_+\right)(\epsilon - 3p) = \frac{2}{\pi} \int_0^\infty \frac{[\rho(\omega)]_T}{\omega} d\omega. \quad (1)$$

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In the limit when the operator O becomes marginal, $\Delta_+ \rightarrow 4$, the sum rule coincides with the sum rule in QCD, derived in [15].

We shall also find that in this limit, which we shall refer to as marginal, the sum rule in Eq. (1) exhibits the same puzzle as discussed in Refs.[15, 33, 34] in the context of QCD: the l.h.s. of the sum rule is of order α_s^3 , while $\rho \sim \alpha_s^2$, where α_s is the QCD coupling. Here, in the marginal limit, the l.h.s. of Eq.(1) is of order Δ_-^3 while $\rho \sim \Delta_-^2$, where

$$\Delta_- = 4 - \Delta_+. \quad (2)$$

We shall demonstrate that a delicate cancellation indeed occurs between the high frequency tail of the integral, $\omega \gg T$, and the intermediate region of $\omega \sim T$, of the kind needed to occur in QCD. Keeping only the leading $\mathcal{O}(\Delta_-^2)$ terms in $[\rho]_T$, we find that the integral in Eq. (1) converges. However, the integral over the contribution of the subleading $\mathcal{O}(\Delta_-^3)$ terms in $[\rho]_T$ has support over an interval of ω which stretches to infinity as $1/\Delta_-$ in the marginal limit $\Delta_- \rightarrow 0$. We evaluate the resulting additional $\mathcal{O}(\Delta_-^2)$ contribution from this long tail to the r.h.s. of Eq. (1) analytically and show that it cancels the $\mathcal{O}(\Delta_-^2)$ contribution from the $\omega \sim T$ region.

This paper is organized as follows. In the next section, we present the definitions of quantities involved in the sum rule and its derivation. In Section III, we derive the sum rule, Eq. (1), in the general class of theories we consider. Section IV introduces the holographic description of such theories. Section V explains our method of calculating the spectral function and the related Green's functions. In Section VI, we make analytical calculation of the large ω asymptotics of $\rho(\omega)$, which we use in Section VII to subtract from the numerically determined $\rho(\omega)$ to obtain $[\rho(\omega)]_T$. The sum rule is verified numerically in Section VIII for a sample of values of Δ_- . We demonstrate the cancellation required to satisfy the sum rule in the marginal limit in Section VIII. Cross-check of the results with the existing analytical result for bulk viscosity (Ref.[35]) is made in Section X. We conclude in Section XI. Appendix A contains the relevant results from [29], used throughout the paper.

II. DEFINITIONS

We shall consider only the response to homogeneous ($\mathbf{q} = 0$) perturbations and define the spectral function for the trace of the stress-energy tensor $T^{\mu\nu}$, as usual, by

$$\rho(\omega) \equiv -\text{Im } G_R(\omega), \quad \text{with} \quad (3)$$

$$G_R(\omega) \equiv -i \int_0^\infty dt e^{i\omega t} \int d^3\mathbf{x} \langle [T_\mu^\mu(x), T_\nu^\nu(0)] \rangle. \quad (4)$$

The definition of the retarded Green's function G_R is subject to the usual ambiguity due to the contribution of the product of the operators at the same point at $x = 0$. However, these contact terms do not have imaginary parts and do not contribute to ρ .

It is convenient, as in [15], to define Green's functions of $T^{\mu\nu}$ by considering the response of the system to the perturbation of the background metric $g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}$ around the flat Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. Euclidean time correlators can be obtained similarly from variations of the Euclidean partition function $Z[g]$:

$$\log Z[g] = \frac{1}{2} \int d^4x \langle T^{\mu\nu}(x) \rangle \delta g_{\mu\nu}(x) + \frac{1}{8} \iint d^4x d^4y \langle T^{\mu\nu}(x) T^{\lambda\rho}(y) \rangle \delta g_{\mu\nu}(x) \delta g_{\lambda\rho}(y) + \mathcal{O}(g^3). \quad (5)$$

The correlation functions of the trace $\theta = T_\mu^\mu$ can be also defined via variations of the partition function with respect to the metric variations of a special form (dilations):

$$g(\Omega)_{\mu\nu} = \eta_{\mu\nu} e^{-2\Omega} \quad (6)$$

$$\log Z[g(\Omega)] = - \int d^4x \langle \theta(x) \rangle \Omega(x) + \frac{1}{2} \iint d^4x d^4y \langle \theta(x) \theta(y) \rangle \Omega(x) \Omega(y) + \mathcal{O}(\Omega^3). \quad (7)$$

The two definitions of the two-point function of the trace differ by a contact term:

$$\langle \theta(x) \rangle = \eta_{\mu\nu} \langle T^{\mu\nu}(x) \rangle \quad (8)$$

$$\langle \theta(x) \theta(y) \rangle = \eta_{\mu\nu} \eta_{\lambda\rho} \langle T^{\mu\nu}(x) T^{\lambda\rho}(y) \rangle + 2\delta^4(x-y) \eta_{\mu\nu} \langle T^{\mu\nu}(x) \rangle \quad (9)$$

The corresponding retarded correlators can be defined via linear response to perturbation Ω :

$$G_R(x, y) = \langle \theta(x) \theta(y) \rangle_R = \frac{\partial}{\partial \Omega(x)} \left(\sqrt{-g(\Omega(y))} \langle \theta(y) \rangle \right) \quad (10)$$

The definition (10) is convenient because the value of its Fourier transform (for $\mathbf{q} = 0$)

$$G_R(\omega) = \langle \theta \theta \rangle_R(\omega) \equiv \int dt e^{i\omega t} \int d^3 \mathbf{x} \langle \theta(x) \theta(0) \rangle_R \quad (11)$$

at vanishing frequency ω follows from the conservation of entropy in the ideal hydrodynamics [15]:

$$\langle \theta \theta \rangle_R(0) = \frac{\partial}{\partial \Omega} \left(\sqrt{-g(\Omega)} \langle \theta \rangle \right) = \left(3s \frac{\partial}{\partial s} - 4 \right) \langle \theta \rangle \quad (12)$$

where s is the entropy density.

III. THE SUM RULE

The sum rule derived in [15] applies to the zero temperature subtracted Green's function

$$[G_R]_T \equiv G_R - G_R^{(T=0)} \quad (13)$$

and its imaginary part on the real axis $[\rho]_T = -\text{Im}[G_R]_T$:

$$[G_R(i\infty)]_T - [G_R(0)]_T = \frac{2}{\pi} \int_0^\infty d\omega \frac{[\rho(\omega)]_T}{\omega}. \quad (14)$$

Using Eq. (12) and $[\langle \theta \rangle]_T = 3p - \epsilon$, one can write

$$[G_R(0)]_T = - \left(3s \frac{\partial}{\partial s} - 4 \right) (\epsilon - 3p) \quad (15)$$

where $\epsilon = [T^{00}]_T$ and $p = [T^{11}]_T$ are equilibrium thermal energy and pressure at given entropy density s . In QCD, due to the asymptotic freedom, $[G_R(i\infty)]_T = 0$ and one obtains the sum rule found in [15]:

$$\left(3s \frac{\partial}{\partial s} - 4 \right) (\epsilon - 3p) = \frac{2}{\pi} \int d\omega \frac{[\rho(\omega)]_T}{\omega} \quad (16)$$

In this paper, we consider a generic conformal field theory perturbed by an operator O of dimension Δ_+ sourced by the field c . We place this theory on a nontrivial gravitational background given by metric $g_{\mu\nu}$. The change of the partition function of the theory under dilatations (6) is equivalent to the rescaling of the only dimensionful external field c , the source of the operator O :

$$Z[g(\Omega), c] = Z[g(0), e^{-\Delta_- \Omega} c], \quad (17)$$

where $\Delta_- = 4 - \Delta_+$. Thus

$$\frac{\delta \log Z}{\delta \Omega} = -\Delta_- e^{-\Delta_- \Omega} c \frac{\delta \log Z}{\delta c} = \Delta_- e^{-\Delta_- \Omega} c \langle O \rangle. \quad (18)$$

Thus

$$\langle \theta(x) \rangle = -\Delta_- c \langle O(x) \rangle, \quad (19)$$

and

$$\langle \theta(x) \theta(y) \rangle = \Delta_-^2 c^2 \langle O(x) O(y) \rangle - \Delta_-^2 c \langle O \rangle \delta^4(x - y). \quad (20)$$

To evaluate the Fourier transform of the correlation function (20) at large (imaginary) frequency ω , we can use the operator product expansion (OPE) for $O(x)O(y)$. The leading contribution comes from the unit operator, and behaves as $\omega^{2\Delta_+ - 4}$, but cancels in $[G_R]_T$ because it is independent of temperature. The contribution of an operator of dimension Δ comes with the Wilson coefficient which behaves as $\omega^{2\Delta_+ - 4 - \Delta}$. Assuming that there are no operators (with vacuum quantum numbers) of dimension equal to or lower than $2\Delta_+ - 4$, we conclude that the contribution of the first term in Eq. (20) vanishes in the $\omega \rightarrow i\infty$ limit. The assumption implies, in particular, that $\Delta_+ < 4$, i.e., $\Delta_- > 0$. The contribution of the last term in Eq. (20) then gives the value for $G_R(i\infty)$:

$$[G_R(i\infty)]_T = -[\langle \theta \theta \rangle(i\infty)]_T = \Delta_-^2 c [\langle O \rangle]_T = \Delta_- (\epsilon - 3p), \quad (21)$$

where we used the fact that analytical continuation of a retarded correlation function to Matsubara frequencies on the imaginary axis equals negative of the Euclidean correlator. We shall verify Eq. (21) explicitly and analytically in the holographic model. Combining Eqs. (15), (21) and (14) we obtain Eq. (1).

IV. HOLOGRAPHIC MODEL

As discussed above, we will consider a four-dimensional (4D) conformal theory in which the conformal symmetry is broken by the operator O with scaling dimension Δ_+ . The holographically dual description of such a theory must therefore include a five-dimensional (5D) scalar field with mass $m_5^2 = -\Delta_+\Delta_-$. As a model for the QCD thermodynamics, such a theory was first considered in [26, 27]. It models in the most straightforward way the breaking of the scaling invariance in QCD. The minimum set of fields required to describe correlators of the stress energy tensor and of the operator O include the 5D metric g_{MN} ($M, N = 0, 1, 2, 3, z$) and the scalar field (dilaton) ϕ . The minimal action is thus given by

$$S_5 = S_{\text{bulk}} + S_{\text{GH}} = \frac{1}{2\kappa^2} \left[\int_M d^5x \sqrt{-g} \left(R - V(\phi) - \frac{1}{2} (\partial\phi)^2 \right) - 2 \int_{\partial M} d^4x \sqrt{-\gamma} K \right], \quad (22)$$

where R is the Ricci scalar, g is the determinant of the metric, γ is the determinant of the induced metric on the UV boundary ∂M at $z = 0$, K is the extrinsic curvature on ∂M , and κ^2 is the 5D Einstein gravitational constant. The value of κ^2 is inversely proportional to the number of the degrees of freedom in the dual four-dimensional theory, e.g., N_c^2 in a gauge theory with large number of colors N_c . The smallness of κ^2 (i.e., the largeness of the number of colors) controls the semiclassical approximation which we use. The last term in the action is the Gibbons-Hawking term, which removes the boundary terms arising upon integration by parts of the terms in R linear in second derivatives of the metric [36].

The potential for the dilaton, $V(\phi)$, is the function which was tuned in [26, 27] to best “mimic” the QCD equation of state. Here, similarly to [29], we shall concentrate on the results which are universal in the class of models described by Eq. (22) with any (sensible) potential. For example, Ref. [29] found that the speed of sound approaches the conformal value $1/\sqrt{3}$ as $T \rightarrow \infty$ universally from below in such theories. Here we shall also make use of the large T limit to the extent that it makes only the curvature of the potential $V''(\phi) = m_5^2 = \Delta_+(\Delta_+ - 4)$ matter.

The negative cosmological constant provided by choosing $V(0) = -12$ ensures necessary asymptotics of the metric near the boundary $g_{MN} \sim z^{-2}$ while the dilaton field asymptotics is fixed by $V''(\phi)$: $\phi \sim z^{\Delta_-}$. The singular behavior at $z = 0$ can be regulated by setting the boundary conditions at $z = \varepsilon$ and taking the UV regulator ε to 0 after necessary renormalization. The rules of the holographic correspondence require us to extremize the action S_5 w.r.t. the metric and the dilaton field subject to the boundary conditions on the UV boundary $z = \varepsilon \rightarrow 0$:

$$g_{\mu\nu}(x, z)|_{z=\varepsilon} = g_{\mu\nu}(x) \varepsilon^{-2}, \quad \phi(x, z)|_{z=\varepsilon} = c(x) \varepsilon^{\Delta_-}. \quad (23)$$

The holographic duality then implies that the extremal value of the Euclidean 5D action as a functional of the boundary values $g_{\mu\nu}(x)$ and $c(x)$ is the same functional as $-\log Z[g_{\mu\nu}, c]$ in the 4D theory. In order to calculate the Green’s function $\langle \theta\theta \rangle$ we use Eq. (9) and the relationship (5) between the derivatives of $\log Z$ and the correlators of $T^{\mu\nu}$.

The one-point functions $\langle T^{\mu\nu} \rangle$ are determined by the solutions of the equations of motion with boundary conditions homogeneous in the 4D coordinates x^λ . We follow Ref. [29] to determine the solution to the equations of motion homogeneous in 4D. In the gauge chosen in Ref. [29], the metric has the form

$$ds^2 = \frac{1}{z^2} \left(-f(z) dt^2 + d\mathbf{x}^2 + e^{2B(z)} \frac{dz^2}{f(z)} \right) \equiv g_{MN}^{(0)} dx^M dx^N, \quad (24)$$

where $f(z)$ and $B(z)$ are functions only of the extra dimensional coordinate z , and $f(z)$ has a simple zero at some value of $z = z_H$. The functions, f and B , and the background dilaton field, ϕ , are all determined by extremizing the action. The details can be found in [29] and for completeness presented in Appendix A.

The extremum of the action with boundary conditions (23) is a one-parameter family of solutions to the equations of motion. A convenient choice of the parameter is the enthalpy density, w , which arises as an integration constant (see Eq. (A6)). The temperature is determined, as usual, by considering the periodicity in the Euclidean time necessary to avoid conical singularity at $z = z_H$:

$$T = \frac{1}{4\pi} e^{-B(z_H)} |f'(z_H)|, \quad (25)$$

and is related to w via equation of state (see e.g. Eq. (A15)).

V. TWO-POINT FUNCTIONS

The calculation of two-point correlation functions requires us to consider non-homogeneous solutions to the equations of motion. We only need to consider infinitesimal variations around the homogeneous solution (24) and expand the action S_5 to quadratic order in these variations. We parameterize these inhomogeneous solutions as

$$g_{MN} = g_{MN}^{(0)}(1 + h_{MN}) \quad (26)$$

where summation over M, N is not implied. We exercise the freedom of the gauge choice to set $\phi(x, z) = \phi^{(0)}(z)$, i.e., we set variations of ϕ around homogeneous solution to 0 (we can do this as long as we are not interested in calculating correlation functions of O , i.e., as long as c remains homogeneous). We find that this gauge choice provides nontrivial simplification of the equations of motion, allowing us to reduce them to a second order equation for metric variation H , instead of the third.

We then consider solution homogeneous in the spatial coordinates x^i , and depending only on $x^0 \equiv t$ and z , since we are interested in $\mathbf{q} = 0$ variations of $g_{\mu\nu}$. For such solutions we can use the remaining gauge freedom to set $h_{z\mu} = 0$, but not h_{zz} .

We can also use $O(3)$ symmetry to simplify our analysis by separating the spatial part of metric perturbations into trace and traceless parts:

$$h_{ij} = H\delta_{ij} + h_{ij}^T. \quad (27)$$

At quadratic order, H mixes with components h_{00} and h_{zz} , but decouples from the traceless part h_{ij}^T as well as from the off-diagonal components h_{0i} . Therefore, we shall focus only on the metric perturbations H , h_{00} and h_{zz} . We can express the coupling of the metric perturbation to $T^{\mu\nu}$ as

$$h_{\mu\nu}T^{\mu\nu} = H\Sigma + h_{ij}^T T_T^{ij} + 2h_{0i}T^{0i} + h_{00}T^{00}, \quad (28)$$

where $\Sigma = T^{ij}\delta_{ij}$ is the trace and $T_T^{ij} = T^{ij} - \frac{1}{3}\Sigma\delta^{ij}$ is the traceless part of the *stress* tensor T^{ij} . The two-point function $\langle\theta\theta\rangle$ in Eq. (9), since $\eta_{\mu\nu}T^{\mu\nu} = \Sigma - T^{00}$, can be expressed in terms of correlation functions of T^{00} and Σ using

$$\eta_{\mu\nu}\eta_{\lambda\rho}\langle T^{\mu\nu}(x)T^{\lambda\rho}(y)\rangle = \langle T^{00}(x)T^{00}(y)\rangle - 2\langle T^{00}(x)\Sigma(y)\rangle + \langle\Sigma(x)\Sigma(y)\rangle. \quad (29)$$

The corresponding retarded correlators can be found using holographic correspondence and the recipe [37]:

$$\langle T^{00}(t)T^{00}(t')\rangle_R = -4 \left. \frac{\delta^2 S_5}{\delta h_{00}(t, z)\delta h_{00}(t', z)} \right|_{z=\varepsilon}, \quad (30)$$

$$\langle T^{00}(t)\Sigma(t')\rangle_R = -4 \left. \frac{\delta^2 S_5}{\delta h_{00}(t, z)\delta H(t', z)} \right|_{z=\varepsilon}, \quad (31)$$

$$\langle\Sigma(t)\Sigma(t')\rangle_R = -4 \left. \frac{\delta^2 S_5}{\delta H(t, z)\delta H(t', z)} \right|_{z=\varepsilon}. \quad (32)$$

Since we need the Fourier transform of $\langle\theta\theta\rangle$ it would be convenient to express S_5 directly in terms of the Fourier modes of $h_{\mu\nu}$ defined as

$$h_{MN}(t, z) = \int \frac{d\omega}{2\pi} h_{MN}(\omega, z) e^{-i\omega t}. \quad (33)$$

The metric perturbations H , h_{zz} and h_{00} , defined by Eqs. (26) and (27), satisfy a set of equations derived from the linearized Einstein's equations:

$$H''(\omega, z) = H'(\omega, z) \left(\frac{2}{z} + B'(z) - \frac{f'(z)}{f(z)} - \frac{B''(z)}{B'(z)} \right) + H(\omega, z) \left(-\frac{\omega^2 e^{2B(z)}}{f(z)^2} + \frac{f'(z)}{2zf(z)} + \frac{f'(z)}{2f(z)} \frac{B''(z)}{B'(z)} \right), \quad (34)$$

$$h'_{00}(\omega, z) = H'(\omega, z) \left(-1 - zB'(z) + \frac{z}{2} \frac{f'(z)}{f(z)} \right) + H(\omega, z) \left(2 \frac{f'(z)}{f(z)} + \frac{z}{2} \frac{f'(z)B'(z)}{f(z)^2} - \omega^2 \frac{z e^{2B(z)}}{f(z)^2} - \frac{z}{2} \frac{f'(z)^2}{f(z)^2} \right), \quad (35)$$

$$h_{zz}(\omega, z) = -zH'(\omega, z) + H(\omega, z) \left(\frac{z}{2} \frac{f'(z)}{f(z)} \right), \quad (36)$$

with the primes denoting derivatives with respect to z . In order for Eqs. (30)–(32) to give the *retarded* correlation functions, the solutions to Eqs. (34)–(36) must satisfy the in-falling wave conditions at the horizon $z = z_H$.

In order to calculate the two-point functions, we expand the holographic action in Eq. (22) to quadratic order in metric variations around the homogeneous background. This expansion can be written in a compact form; the bulk part of the action is given by,

$$S_{\text{bulk}} = \frac{1}{2\kappa^2} \int \frac{d\omega}{2\pi} d^3x dz \frac{e^{-B(z)} f(z)}{z^3} (\mathbf{H}'^\dagger M_1 \mathbf{H} + \mathbf{H}'^\dagger M_2 \mathbf{H}' + \mathbf{H}'^\dagger M_3 \mathbf{H} + \mathbf{H}^\dagger M_4 \mathbf{H} + (\text{c.c.})), \quad (37)$$

where the dagger refers to the transposed complex conjugate ($\mathbf{H}^*(\omega, z) = \mathbf{H}(-\omega, z)$), and

$$\begin{aligned} \mathbf{H} &= \begin{pmatrix} H(\omega, z) \\ h_{00}(\omega, z) \\ h_{zz}(\omega, z) \end{pmatrix}; \quad M_1 = \frac{1}{2} \begin{pmatrix} -3 & 3 & 3 \\ 3 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}; \quad M_2 = \frac{1}{4} \begin{pmatrix} 0 & 3 & 3 \\ 3 & 2 & -1 \\ 3 & -1 & 0 \end{pmatrix}; \\ M_3 &= \frac{2}{z} \begin{pmatrix} 3 & -3 & -3 \\ -3 & -1 & 1 \\ -3 & 1 & 3 \end{pmatrix} + \frac{1}{2} B'(z) \begin{pmatrix} 3 & -3 & -3 \\ -3 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{4} \frac{f'(z)}{f(z)} \begin{pmatrix} -6 & 6 & 6 \\ 9 & 3 & -3 \\ 3 & -1 & -3 \end{pmatrix}; \\ M_4 &= \frac{1}{4z} \left(\frac{4}{z} + B'(z) - \frac{f'(z)}{f(z)} \right) \begin{pmatrix} -3 & 3 & 12 \\ 3 & 1 & -4 \\ 12 & -4 & -9 \end{pmatrix} - \frac{3\omega^2 e^{2B(z)}}{4f(z)^2} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (38)$$

Integrating by parts and using equation of motion, we reduce the action at the extremum to the boundary term:

$$S_{\text{bulk}} = -\frac{1}{2\kappa^2} \int \frac{d\omega}{2\pi} d^3x \frac{e^{-B(z)} f(z)}{z^3} \left(\mathbf{H}'^\dagger M_1 \mathbf{H} - \mathbf{H}^\dagger M_1 \mathbf{H} \left(\frac{f'(z)}{f(z)} - B'(z) - \frac{3}{z} \right) - \mathbf{H}^\dagger M_1 \mathbf{H}' + 2\mathbf{H}^\dagger M_2 \mathbf{H}' + \mathbf{H}^\dagger M_3 \mathbf{H} \right) \Big|_{z=\varepsilon}. \quad (39)$$

The action also receives a contribution from the Gibbons-Hawking boundary term:

$$S_{\text{GH}} = \frac{1}{2\kappa^2} \int \frac{d\omega}{2\pi} d^3x \frac{e^{-B(z)} f(z)}{z^3} (\mathbf{H}'^\dagger M_5 \mathbf{H} + \mathbf{H}^\dagger M_6 \mathbf{H} + (\text{c.c.})) \Big|_{z=\varepsilon}; \quad \text{with} \quad (40)$$

$$M_5 = M_1 \quad \text{and} \quad M_6 = \frac{1}{8} \left(\frac{8}{z} - \frac{f'(z)}{f(z)} \right) \begin{pmatrix} 3 & -3 & -3 \\ -3 & -1 & 1 \\ -3 & 1 & 3 \end{pmatrix}.$$

Combining all of this, the action at the extremum can be expressed in terms of the boundary values of h_{00} , H and H' :

$$\begin{aligned} S_5 &= \frac{1}{2\kappa^2} \int \frac{d\omega}{2\pi} d^3x \frac{e^{-B(z)} f(z)}{z^3} \left(-\frac{3}{2z} h_{00}^*(\omega, z) h_{00}(\omega, z) + \left(-\frac{9}{2z} + \frac{3}{4} \frac{f'(z)}{f(z)} \right) (h_{00}^*(\omega, z) H(\omega, z) + (\text{c.c.})) \right. \\ &\quad + \left(\frac{9}{2z^2} + \frac{3}{4} \frac{B'(z) f'(z)}{f(z)} - \frac{3}{8} \frac{f'(z)^2}{f(z)^2} - \frac{3\omega^2 e^{2B(z)}}{2f(z)^2} \right) z H^*(\omega, z) H(\omega, z) \\ &\quad \left. - \frac{3}{2} z B'(z) H^*(\omega, z) H'(\omega, z) \right) \Big|_{z=\varepsilon}. \end{aligned} \quad (41)$$

From this one can immediately read off the expressions for the Fourier transforms of the stress-energy two-point functions:

$$\langle T^{00} T^{00} \rangle(\omega) = \frac{1}{2\kappa^2} \frac{6e^{-B(z)} f(z)}{z^4} \Big|_{z=\varepsilon} = -\langle T^{00} \rangle; \quad (42)$$

$$\langle T^{00} \Sigma \rangle(\omega) = \frac{3}{2\kappa^2} \left(\frac{6e^{-B(z)} f(z)}{z^4} - \frac{e^{-B(z)} f'(z)}{z^3} \right) \Big|_{z=\varepsilon} = \langle \Sigma \rangle; \quad (43)$$

$$\langle \Sigma \Sigma \rangle(\omega) = \frac{9}{2\kappa^2} \frac{e^{-B(z)} f(z)}{z^2} \left(-\frac{2}{z^2} - \frac{B'(z) f'(z)}{3f(z)} + \frac{f'(z)^2}{6f(z)^2} + \frac{2\omega^2 e^{2B(z)}}{3f(z)^2} + \frac{2B'(z)}{3} \frac{H'(\omega, z)}{H(\omega, z)} \right) \Big|_{z=\varepsilon}, \quad (44)$$

where in the last equality in Eqs. (42) and (43) we used expressions for one-point functions Ref. [29]. As expected, the correlators involving T^{00} are frequency independent and satisfy energy-momentum conservation Ward identities Ref. [38]. The above expressions contain contributions divergent in the limit $\varepsilon \rightarrow 0$. They can be removed by subtracting a polynomial of ω^2 sufficient to cancel the divergences. These terms do not affect $\rho(\omega)$.

Putting this all together, the finite (as $\varepsilon \rightarrow 0$) part of the bulk Green's function is found to be,

$$G_R(\omega) - P(\omega^2) = 4(\epsilon - 3p) + \frac{6}{2\kappa^2} \frac{e^{-B(z)} f(z) B'(z)}{z^2} \frac{H'(\omega, z)}{H(\omega, z)} \Big|_{z=\varepsilon}, \quad (45)$$

where the polynomial $P(\omega^2) = 4\langle T^{00} - \Sigma \rangle_{T=0} + \omega^2 \cdot 3e^{B(\varepsilon)} / (\kappa^2 \varepsilon^2)$ combines the UV divergent, temperature independent contact terms. The bulk spectral function is then given by

$$\rho(\omega) = -\frac{6}{2\kappa^2} \frac{e^{-B(z)} f(z) B'(z)}{z^2} \text{Im} \frac{H'(\omega, z)}{H(\omega, z)} \Big|_{z=\varepsilon}. \quad (46)$$

VI. LARGE FREQUENCY ASYMPTOTICS

In order to calculate the bulk retarded Green's function and the bulk spectral function, one needs to solve the equation of motion for the field $H(\omega, z)$, Eq. (34). In general, this cannot be done analytically. However, in the large ω limit, the solution can be found similarly to the Born approximation in quantum mechanics.

We perform a Louville transformation to bring Eq. (34) to the Schrödinger form. In terms of the new coordinate x , given by

$$x = \int^z \frac{e^{B(z')}}{f(z')} dz', \quad (47)$$

and the new function $\Psi(x)$, given by

$$H(z) = z B'(z)^{-1/2} \Psi(x), \quad (48)$$

Eq. (34) takes the form

$$\frac{d^2 \Psi(x)}{dx^2} + \Psi(x) (\omega^2 - U_{\text{Sch}}(x)) = 0, \quad (49)$$

with the Schrödinger potential, as an implicit function of x , is given by

$$U_{\text{Sch}}(x) = \frac{f(z)^2}{e^{2B(z)}} \left(\frac{2}{z^2} - \frac{1}{z} \frac{B''(z)}{B'(z)} - \frac{1}{4} \frac{B''(z)^2}{B'(z)^2} + \frac{1}{2} \frac{B'''(z)}{B'(z)} - \frac{1}{2z} \frac{f'(z)}{f(z)} + \frac{f'(z)}{f(z)} \frac{B''(z)}{B'(z)} + \frac{1}{z} B'(z) - \frac{1}{2} B''(z) \right). \quad (50)$$

The in-falling boundary condition on $H(z)$ at the horizon $z = z_H$ transforms into the outgoing wave condition on $\Psi(x)$ at $x = \infty$. The bulk spectral function can be expressed in terms of the new function Ψ as

$$\rho = -\frac{1}{2\kappa^2} \frac{6B'(z)}{z^2} \text{Im} \frac{\Psi'(x)}{\Psi(x)} \Big|_{z=\varepsilon}, \quad (51)$$

or somewhat more intuitively (using Eq. (A1)) as

$$\rho = \frac{\phi'(z)^2}{2\kappa^2 z} \frac{\text{Im} [\Psi^*(x) \Psi'(x)]}{|\Psi(x)|^2} \Big|_{z=\varepsilon} \quad (52)$$

which has the quantum mechanical interpretation of the probability flux for the wave function Ψ (normalized as $|\Psi(x)|^2 = \phi'(\varepsilon)^2 / (2\kappa^2 \varepsilon)$ at $z = \varepsilon$). The positivity of the spectral function can be seen then as a consequence of the conservation of the flux and the outgoing wave boundary condition at $x = \infty$.

At large ω , we would like to treat the potential as a perturbation and calculate the “wave function” Ψ using the Born approximation. There is one difficulty, however: the Schrödinger potential $U_{\text{Sch}}(x)$ diverges at $x = 0$. We shall separate the leading divergence explicitly

$$\frac{d^2 \Psi(x)}{dx^2} + \Psi(x) \left(\omega^2 - \frac{(3 - 2\Delta_-)(2\Delta_+ - 3)}{4x^2} - \delta U_{\text{Sch}}(x) \right) = 0 \quad (53)$$

and treat the remaining part of the potential, $\delta U_{\text{Sch}}(x)$, as a perturbation. We can calculate $\Psi(x)$ iteratively, using the Green's function method. The solution to Eq. (53) with $\delta U_{\text{Sch}} = 0$, satisfying the outgoing wave boundary condition at $x = \infty$, is given by, up to unimportant normalization,

$$\Psi_0(x) = (\omega x)^{1/2} H_\nu^{(1)}(\omega x) \quad (54)$$

where $\nu = 2 - \Delta_-$, $H_\nu^{(1)}$ is the Hankel function the first kind. Substituting Ψ_0 into (48) and then into (45) we find, using Eq. (A9) for $B'(z)$,

$$G_R(\omega) - P(\omega^2) = \frac{c^2}{2\kappa^2} \frac{2\pi\Delta_-^2}{\Gamma(2-\Delta_-)^2} (\cot(\pi\Delta_-) - i) \left(\frac{\omega}{2}\right)^{\Delta_+ - \Delta_-} - \frac{cd}{2\kappa^2} \Delta_-^2 (\Delta_+ - \Delta_-) + \dots \quad (55)$$

where the dots stand for corrections to the leading order which would come from iterations of δU_{Sch} . The first term on the r.h.s. is the leading term in the $\omega \rightarrow \infty$ limit. It is temperature independent and has correct ω -scaling to be identified with the leading contribution of the unit operator to the OPE of $\langle OO \rangle$ in Eq. (20). This term has nontrivial, but temperature independent, imaginary part and it is subtracted when we calculate $[G_R]_T$.

The second term on the r.h.s. of Eq. (55) gives

$$[G_R(i\infty)]_T = -\frac{c[d]_T}{2\kappa^2} \Delta_-^2 (\Delta_+ - \Delta_-) = \Delta_- (\epsilon - 3p). \quad (56)$$

where we used Eq. (A11). This agrees with Eq. (21) as expected.

The leading correction, Ψ_1 , to Ψ_0 can be found, similarly to the Born approximation, by using Green's function $G(x, x')$:

$$\Psi_1(x) = \int dx' G(x, x') \Psi_0(x') \delta U_{\text{Sch}}(x'). \quad (57)$$

The Green's function satisfies equation

$$\frac{d^2 G(x, x')}{dx^2} + G(x, x') \left(\omega^2 - \frac{(2\Delta_- - 3)(2\Delta_+ - 3)}{4x^2} \right) = \delta(x - x'), \quad (58)$$

with boundary condition $G(x, x') = 0$ at $x = \varepsilon$ and outgoing wave condition at $x = \infty$. The solution is given by

$$G(x, x') = \begin{cases} \gamma_{<}(x') (\omega x)^{1/2} J_\nu(\omega x) & x < x' \\ \gamma_{>}(x') (\omega x)^{1/2} H_\nu^{(1)}(\omega x) & x > x' \end{cases} \quad (59)$$

where

$$\gamma_{<}(x') = -\frac{i\pi}{2\omega} (\omega x')^{1/2} H_\nu^{(1)}(\omega x') \quad \text{and} \quad \gamma_{>}(x') = -\frac{i\pi}{2\omega} (\omega x')^{1/2} J_\nu(\omega x'). \quad (60)$$

Substituting into Eq. (57) we find

$$\Psi_1(x) = -\frac{i\pi}{2} (\omega x)^{1/2} \left[J_\nu(\omega x) \int_x^\infty x' \delta U_{\text{Sch}}(x') \left(H_\nu^{(1)}(\omega x') \right)^2 dx' + H_\nu^{(1)}(\omega x) \int_\varepsilon^x x' \delta U_{\text{Sch}}(x') H_\nu^{(1)}(\omega x') J_\nu(\omega x') dx' \right]. \quad (61)$$

To zeroth order in δU_{Sch} , $\Psi = \Psi_0$ and Eq. (51) gives:

$$\rho_0(\omega) = \frac{c^2}{2\kappa^2} \frac{2\pi\Delta_-^2}{\Gamma(2-\Delta_-)^2} \left(\frac{\omega}{2}\right)^{4-2\Delta_-}. \quad (62)$$

To next order in δU_{Sch} , $\Psi = \Psi_0 + \Psi_1$ and $\rho = \rho_0 + \rho_1$, where

$$\rho_1(\omega) = \pi \rho_0(\omega) \int_0^\infty x \delta U_{\text{Sch}}(x) J_\nu(\omega x) Y_\nu(\omega x) dx. \quad (63)$$

In the limit $\omega \rightarrow \infty$, the integral in Eq. (63) is dominated by the region of small $x \sim \omega^{-1}$. Expanding δU_{Sch} in powers of x , we generate a $1/\omega$ series (possibly asymptotic) for the integral.

To enable further analytic calculation, we shall limit it to leading terms in the expansion in powers of c^2 . As we also discuss in Appendix A, this corresponds to the regime of high temperatures, i.e., $T \gg c^{1/\Delta_-}$, since c is the only dimensionful parameter in the theory. In addition to enabling analytic calculations, this limit has the virtue of yielding results which do not depend on the form of the dilaton potential beyond its curvature at the minimum (related to Δ_-). In this regime, to the order we work, we can set c to zero in U_{Sch} . The leading term in the Taylor expansion of U_{Sch} is given by

$$\delta U_{\text{Sch}}(x) = \frac{3}{20} \Delta_- \Delta_+ \bar{w} x^2 + \mathcal{O}(x^6), \quad (64)$$

where $\bar{w} = 2\kappa^2 w$, as defined in Appendix A, and

$$\rho_1 = -\frac{1}{20} \frac{\bar{w}}{\omega^4} \Delta_- \Delta_+ (1 - \Delta_-)(\Delta_+ - 1)(\Delta_+ - \Delta_-) \rho_0(\omega). \quad (65)$$

Combining Eq. (62) and Eq. (65) we see that $\rho_1 \sim \omega^{-2\Delta_-} \rightarrow 0$ as $\omega \rightarrow \infty$.

We note that, strictly speaking, not all $\mathcal{O}(c^2)$ corrections to this result are negligible, since some of them grow with ω . However, as one can check, those corrections do not depend on w (i.e., on T) and are part of the $\mathcal{O}(c^2 \omega^{-2\Delta_-})$ corrections to the Wilson coefficient of the unit operator in the OPE of $\langle OO \rangle$. These terms would be also subtracted if $[\rho(\omega)]_T$ was calculated to the next order in c^2 .

VII. CALCULATING THE SPECTRAL FUNCTION

In order to calculate the bulk spectral function, one needs to first solve the equation of motion for the functions $f(z)$, $B(z)$ and $\phi(z)$ and then use those results to solve Eq. (34) for $H(z)$. This cannot be done analytically for an arbitrary potential $V(\phi)$. However, the first step of this calculation can be done analytically in the limit where the temperature is large compared with the conformal breaking scale, $T \gg c^{1/\Delta_-}$. Another advantage of this limit is that the results are universal in the sense that the only property of the potential $V(\phi)$ which matters is the curvature at the minimum, which determines the dimension of the operator Δ_+ .

Equation (34) would still need to be solved numerically, however. We use a shooting procedure starting from the IR boundary, i.e., the horizon $z = z_H$. The equation has a regular singular point at $z = z_H$. A Fröbenius expansion around this point can be constructed iteratively. Among the two linearly independent solutions the in-falling wave is chosen. The expansion is used to calculate H and H' at a point near the IR boundary. These values are then used as the initial conditions to integrate the differential equation numerically all the way to the UV boundary $z = \varepsilon$. The spectral function can then be determined from the numerical results using Eq. (46).

As a representative example, in Fig. 1(a), we plot the spectral function at order c^2 for $\Delta_- \rightarrow 0$. As expected, it is non-negative and it diverges for large ω . The subtracted spectral function $[\rho(\omega)]_T$ to order $\mathcal{O}(c^2)$ is obtained by subtracting ρ_0 determined analytically in Eq. (62). Figures 1(b) and 2 show $[\rho(\omega)]_T/\omega$ plotted at different values of Δ_- , while the dashed line corresponds to the analytic expressions for the large ω behavior given by ρ_1 in Eq. (65).

It is interesting to compare these plots with the lattice calculations of the spectral function in Fig. 5 of Ref. [12]. In both cases one finds the sign oscillation of the spectral function in the region $\omega \sim 2\pi T$, which is absent in the weak coupling (Boltzmann equation) result [33, 39].

VIII. VERIFYING THE SUM RULE

In the high-temperature limit $T \gg c^{1/\Delta_-}$, the left-hand side (thermodynamic side) of the sum rule Eq. (1) can be analytically calculated in the holographic model. The trace anomaly is given by Eq. (A14), while the derivative with respect to entropy can be expressed via derivative with respect to enthalpy w . Using the expression for d in the high T limit, Eq. (A19), the derivative can be performed, and the left-hand side can be expressed as,

$$\left(3s \frac{\partial}{\partial s} - \Delta_+ \right) (\epsilon - 3p) = \frac{cd}{2\kappa^2} \Delta_-^2 (\Delta_+ - \Delta_-). \quad (66)$$

The right-hand side can be numerically calculated from $[\rho]_T$. For efficiency, we make use of the analytical result for the asymptotics of $[\rho]_T$ Eq. (65). We calculate numerically only the contribution to the integral up to some relatively large frequency ω_{max} . The contribution from ω_{max} to ∞ is calculated analytically using asymptotics $[\rho]_T \approx \rho_1$ from Eq. (65). We chose $\omega_{\text{max}} = 10\pi T$. Table I shows that the sum rule holds to at least within a fraction of a percent.

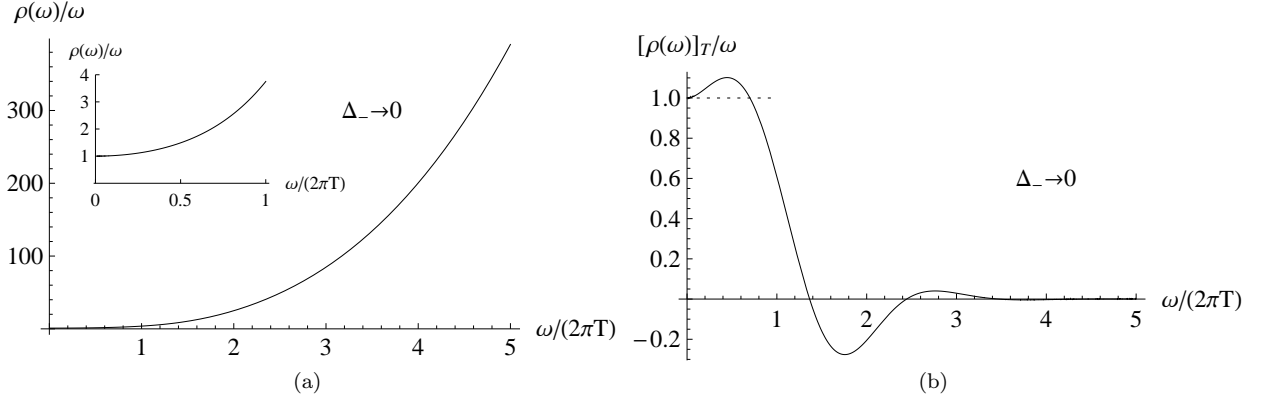


FIG. 1: The bulk spectral function divided by frequency in units of $\frac{1}{2\kappa^2}c^2\Delta_-^2(\pi T)^3$ for $\Delta_- \rightarrow 0$. a) The un-subtracted function. b) The function after the $T = 0$ subtraction.

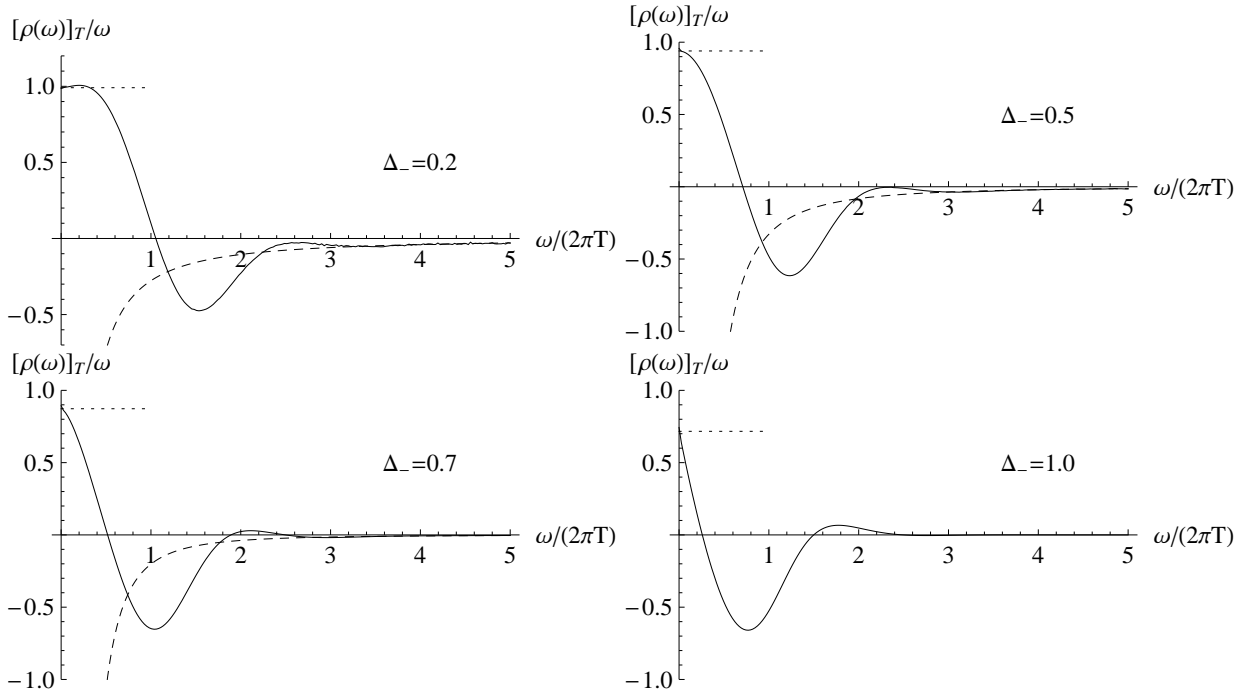


FIG. 2: The bulk spectral function divided by frequency expressed in units of $\frac{1}{2\kappa^2}c^2\Delta_-^2(\pi T)^{3-2\Delta_-}$ for Δ_- equaling 0.2, 0.5, 0.7, and 1.0. The dashed curve in each plot is from the analytic expression for the spectral function in the large ω limit in Eq. (65). The dotted line is drawn at the value of the bulk viscosity calculated from Eq. (71) in units of $\frac{1}{18\kappa^2}c^2\Delta_-^2(\pi T)^{3-2\Delta_-}$. This value should be compared with the intercept of the plots at $\omega = 0$.

IX. THE MARGINAL LIMIT AND THE TAIL OF THE SPECTRAL FUNCTION

As $\Delta_- \rightarrow 0$ the left-hand side of the sum rule given by Eq. (66) scales as Δ_-^3 ($d \sim \Delta_-$ according to Eq. (A19)). This is also apparent in Table I.

However, the spectral function $[\rho]_T$ on the right-hand side scales as Δ_-^2 . This comes from factor $B'(z)$ in Eq. (46) which scales as Δ_-^2 according to Eq. (A9). Therefore, in order to determine the $\mathcal{O}(\Delta_-^2)$ part of $\rho(\omega)$ we can simply

Δ_-	LHS	RHS	% Error
1.0	-0.4569	-0.4588	0.4
0.7	-0.4862	-0.4889	0.6
0.5	-0.4951	-0.4984	0.6
0.2	-0.4997	-0.5025	0.5

TABLE I: Verification of the sum rule in Eq. (1) in the holographic model for a sample of values of Δ_- in the high-temperature limit $T \gg c^{1/\Delta_-}$. The columns LHS and the RHS are the values of the left- and the right- hand side of Eq. (1) expressed in units of $\frac{1}{2\kappa^2} c^2 \Delta_-^3 (\pi T)^{3-2\Delta_-}$, which removes the dependence on κ, c, T and the leading (Δ_-^3) dependence on Δ_- . The LHS is calculated analytically using Eq. (66), while the RHS is a numerical calculation of the integral as described in text. The last column shows the percent discrepancy between the two sides.

set $\Delta_- = 0$ in Eq. (34) which then becomes ¹

$$H''(z) = H'(z) \left(\frac{3}{z} - \frac{f'(z)}{f(z)} \right) - \frac{\omega^2}{f(z)^2} H(z), \quad (67)$$

where f is given by Eq. (A16) in the high-temperature limit. This is the well known equation of motion of a massless scalar field on a pure AdS background. However, the analytic solution to this equation with required boundary conditions has not yet been found (see [40–43] for discussion). Nevertheless, the solution can be obtained numerically as discussed in Sec. VIII. The resulting spectral function is displayed in Fig. 1. The $\mathcal{O}(\Delta_-^2)$ part of $[\rho(\omega)]_T$ shown in Fig. 1(b) oscillates around 0, but the contribution from it to the spectral integral in the sum rule is non-vanishing. We calculate the integral numerically and find

$$\frac{2}{\pi} \int_0^{\omega_{\max}} \frac{[\rho(\omega)]_T}{\omega} d\omega = 2c^2 \Delta_-^2 \left(\frac{w}{4} \right) (.60000(1)) + \mathcal{O}(\Delta_-^3), \quad (68)$$

where w is the enthalpy. The integral of the Δ_-^2 part of $[\rho(\omega)]_T$ is converging very fast (apparently exponentially) at $\omega = \infty$ and we picked relatively large $\omega_{\max} = 10\pi T$ for this numerical calculation. The number in parenthesis is $3/5$ to at least four significant digits. How then is the sum rule satisfied if the left-hand side is only $\mathcal{O}(\Delta_-^3)$?

We shall find the missing contribution in the tail of the function $[\rho(\omega)]_T$. Indeed, taking ω_{\max} to ∞ requires extra care. Because $\mathcal{O}(\Delta_-^2)$ part of $[\rho(\omega)]_T$ decreases very fast as $\omega \rightarrow \infty$, for sufficiently large ω the dominant part in $[\rho(\omega)]_T$ is of order Δ_-^3 . As we see analytically in Eqs. (65), (62), $[\rho]_T \rightarrow \rho_1 \sim \Delta_-^3 \omega^{-2\Delta_-}$. At any finite Δ_- the negative power-law tail effectively cuts off the integral at large $\omega_{\text{tail}} \sim \exp(1/(2\Delta_-))$. However, the contribution of this long tail to the integral grows with ω_{tail} as $\int_{\omega_{\max}}^{\omega_{\text{tail}}} d\omega [\rho]_T / \omega \sim \Delta_-^3 \log(\omega_{\text{tail}}/\omega_{\max}) \sim \Delta_-^2$, i.e., the length of the tail compensates for the extra power of Δ_- .

Let us calculate this $\mathcal{O}(\Delta_-^2)$ contribution from the tail. Using Eqs. (65), (62) we can write for large ω and small Δ_- :

$$[\rho(\omega)]_T = \rho_1(\omega) + \dots = -\frac{6}{5} \pi c^2 \Delta_-^3 \omega^{-2\Delta_-} \left(\frac{w}{4} \right) + \dots \quad (69)$$

Thus

$$\frac{2}{\pi} \int_{\omega_{\max}}^{\infty} \frac{d\omega}{\omega} [\rho]_T = -2c^2 \Delta_-^2 \left(\frac{w}{4} \right) \left(\frac{3}{5} \right) + \mathcal{O}(\Delta_-^3). \quad (70)$$

We see that the contribution of the high-frequency tail exactly cancels the contribution from the region of $\omega \sim T$ in Eq. (68). Therefore the RHS of the sum rule is proportional to Δ_-^3 just as the LHS.

Remarkably, this cancellation mechanism is very similar to the one which was found in QCD by Caron-Huot in Ref.[34]. In the case of QCD the left-hand (thermodynamic) side of the sum rule is of order α_s^3 , while the spectral function on the right-hand side is $\mathcal{O}(\alpha_s^2)$, with logarithmically long high-frequency tail of order $\mathcal{O}(\alpha_s^3)$.

We would like also to comment that the Δ_-^2 contribution to the bulk spectral function $[\rho]_T$ in the high-temperature regime is the same, up to a constant, as the spectral function in the $N = 4$ SUSY YM theory. In fact, the spectral integral in Eq. (68) has been also performed numerically in Ref. [15], using a different method and with the same

¹ Note that according to Eq. (A21), $B'(z) \sim c^2$, which means, to $\mathcal{O}(c^2)$ we are working, we can drop terms like $B'(z)$, but not terms like $B''(z)/B'(z) \rightarrow -1/z$.

result. Unlike the case we consider, however, for the shear channel sum rule considered in Ref. [15] that integral saturated the sum rule.

Also worth noting is that the shear and the bulk channel spectral functions are proportional to each other in the Chamblin-Real dilaton model examined in [31] and [44]. In such theories the mechanism of the saturation of the bulk sum rule is the same as that of the shear sum rule and no high/low frequency cancellations are needed in either case.

X. VISCOSITY

As another cross-check of our results, we can compare them to the analytic calculation of the bulk viscosity in the high temperature limit of Ref. [35], which finds

$$\zeta = \frac{1}{9} \frac{c^2 \Delta_-^2}{2\kappa^2} (\pi T)^{3-2\Delta_-} 2^{\Delta_-} \pi \left(\frac{\Gamma(1 - \Delta_-/4)}{\Gamma(1/2 - \Delta_-/4)} \right)^2. \quad (71)$$

This formula was derived by matching gradient expansion of the stress-energy tensor to the gradient expansion of the background metric.

Ideally, one should also be able to derive Eq. (71) by solving Eq. (34) at small ω and using Kubo formula

$$\zeta = \frac{1}{9} \rho'(0). \quad (72)$$

Indeed, this can be done analytically for $\Delta_- \rightarrow 0$. In this case, Eq. (34) reduces to Eq. (67). For small ω , this equation can be easily integrated. Normalizing as $H(0) = 1$ and using in-falling boundary condition at the horizon we find ²

$$H(z) = 1 - \frac{i\omega}{4} \left(\frac{\bar{w}}{4} \right)^{-1/4} \log(1 - \bar{w}z^4/4) + \dots \quad (73)$$

This gives for the spectral function at small ω ,

$$\rho = \frac{c^2 \Delta_-^2}{2\kappa^2} (\pi T)^3 \omega + \dots, \quad (74)$$

and for the bulk viscosity

$$\zeta = \frac{1}{9} \frac{c^2 \Delta_-^2}{2\kappa^2} (\pi T)^3. \quad (75)$$

This agrees with the $\Delta_- \rightarrow 0$ limit of Eq. (71).

Unfortunately, a similar approach at finite Δ_- appears intractable, since an analytic solution to Eq. (34), even at small ω , for arbitrary Δ_- is not known. However, the bulk viscosity can be calculated by applying the Kubo formula Eq. (72) to our numerical results for the spectral function. We have verified the agreement of Eq. (71) with these numerical calculations and illustrated it in Fig. 2.

XI. CONCLUSION

We studied the spectral function corresponding to time-dependent bulk deformation (uniform expansion) in a class of field theories where conformality is broken “softly” in the sense that at high temperature the equation of state approaches conformal limit $\epsilon = 3p$, similar to QCD. The breaking is due to a scalar operator of conformal dimension $\Delta_+ < 4$, which is an analogue of the operator of gluon condensate in QCD. We find that in such theories the bulk spectral function satisfies the sum rule given by Eq. (1).

The sum rule in Eq. (1) is similar to the sum rule in an asymptotically free theory such as QCD derived by Romatschke and Son in Ref. [15]. In the marginal limit $\Delta_+ \rightarrow 4$, the two sum rules are identical. We used this similarity to address an interesting puzzle noted in Ref. [33]. In order to satisfy the sum rule, a delicate cancellation must occur

² Although this solution is not valid all the way to the horizon due to the singularity, for sufficiently small ω it is valid close enough to the horizon to allow matching it to the in-falling wave solution.

between the regions of $\omega \sim T$ and of $\omega \gg T$ in the spectral integral. Ref.[34] showed that this cancellation can be indeed seen in the high-temperature limit of QCD, using weak coupling calculation. Our study of the holographic model shows that such a cancellation is very generic to the whole class of strongly coupled theories with softly broken conformal symmetry.

To make the connection of our model with QCD more tangible, we can observe that in QCD the action contains operator $G_{\mu\nu}G^{\mu\nu}/\alpha_s$ and thus identify the operator O with $G_{\mu\nu}G^{\mu\nu}/\alpha_s$, up to a numerical coefficient. This coefficient can be determined by matching correlation functions (e.g., the bulk spectral function) of the holographic model to QCD, but we shall not need it here. The anomalous dimension of this operator in QCD is given by $\beta(\alpha_s)/\alpha_s = b_0\alpha_s/(2\pi)$. In QCD this anomalous dimension is a function of scale, vanishing logarithmically, i.e., very slowly, with increasing energy-momentum scale. In the class of QCD-like theories we consider, the corresponding quantity is $\Delta_+ - 4 = -\Delta_-$, which is a constant.³ This scaling dimension taken at $\omega = \infty$, determines the value of $[G_R(i\infty)]_T$ in the sum rule (14), according to Eq. (21), and it is responsible for the difference of the general sum rule (1) from QCD.

The holographic theories we consider can serve only as a qualitative or semi-quantitative guide to the QCD phenomena, since QCD becomes a weakly coupled theory at sufficiently high energy-momentum scale due to asymptotic freedom. But this guide might be useful by offering a view complementary to the weak coupling extrapolation out of the domain of asymptotically high energies. The experiments at RHIC provide a powerful argument that the domain of interest in heavy-ion collisions is a strongly coupled domain. It will be very interesting to see to what extent this remains true at LHC energies.

Also notable in this regard are lattice calculations of the spectral function. The striking feature of the lattice results is the sign oscillation of the spectral function in the $\omega \sim 2\pi T$ region. This oscillation is absent in weakly coupled calculations [34, 39], but appears to be a generic feature in the spectral functions obtained in holographic models, as our results in Figs. 1(b) and 2 illustrate. This qualitative difference appears to be another manifestation of the now familiar fact that real-time response (in particular, hydrodynamics) is more sensitive to the coupling strength than, e.g., equation of state.

Acknowledgments

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Appendix A: Background of the holographic model

1. Einstein's equations and one-point functions

This appendix summarizes, for completeness, the relevant results of Ref. [29]. The background geometry corresponding to homogeneous boundary conditions on $g_{\mu\nu}$ and ϕ can be found by solving Einstein's equations for the metric in Eq. (24) which take the form:

$$\dot{B} = -\frac{1}{6}\dot{\phi}^2, \quad (\text{A1})$$

$$\ddot{f} = (4 + \dot{B})\dot{f}, \quad (\text{A2})$$

$$-6\dot{f} + f(24 - \dot{\phi}^2) + 2e^{2B}V(\phi) = 0, \quad (\text{A3})$$

$$\ddot{\phi}f + \dot{\phi}(\dot{f} - f(4 + \dot{B})) - e^{2B}dV(\phi)/d\phi = 0, \quad (\text{A4})$$

where a dot denotes a $\log z$ derivative, e.g., $\dot{\phi} = z d\phi/dz$. The holographic correspondence provides the boundary conditions at the UV boundary $z = \varepsilon$ given by Eqs. (23). Minkowski metric at the boundary requires

$$f(\varepsilon) = 1. \quad (\text{A5})$$

Equation (A2) can be integrated once to give

$$\dot{f} = -\bar{w}z^4e^B. \quad (\text{A6})$$

³ Holographic models which reproduce the effect of the logarithmic running of α_s , have been discussed in, e.g., Refs.[45–47].

The integration constant \bar{w} must be positive if the metric is to possess a horizon $f(z_H) = 0$ at some value of z_H . Since \bar{w} determines the position of the horizon, it is related to temperature, and we find that it is proportional to the enthalpy, Eq. (A12).

A boundary condition on B is not needed because the value of B is determined by Eq. (A3), which is algebraic in B . The role of the second boundary condition for Eq. (A4) is played by the requirement that ϕ is finite at the horizon, $z = z_H$, which is a regular singular point of the second order differential equation (A4).

Near the $z = \varepsilon \rightarrow 0$ boundary, $\phi \rightarrow 0$, $B \rightarrow 0$, $\dot{B} \rightarrow 0$ and $\dot{f} \rightarrow 0$. Equation (A4) for ϕ can be linearized and the asymptotic behavior of ϕ near the boundary can be determined easily:

$$\phi(z) \rightarrow (c - d\varepsilon^{\Delta_+ - \Delta_-}) z^{\Delta_-} (1 + \dots) + d z^{\Delta_+} (1 + \dots), \quad (\text{A7})$$

where the curvature of the potential $V''(0) \equiv m_5^2$ determines the indices $\Delta_{\pm} = 2 \pm \sqrt{4 + m^2}$. The coefficient of the first term is related to c by Eq. (23). The coefficient d of the second linearly independent solution should be determined by the finiteness condition at the horizon and is a function of \bar{w} (i.e., temperature) and c .

By calculating the derivative of the 5D action with respect to c and matching it, by holographic correspondence, to the expectation value $\langle O \rangle$, one finds (see also [48])

$$\langle O \rangle = -\frac{\partial S_5}{\partial c} = -\frac{e^{-B(z)} \phi'(z)}{2\kappa^2 z^{3-\Delta_-}} \Big|_{z=\varepsilon} = -\frac{d}{2\kappa^2} (\Delta_+ - \Delta_-) + \dots, \quad (\text{A8})$$

where “...” denote UV divergent but temperature independent terms. One can thus see that the integration constant d is related to the UV finite (and temperature dependent) part of $\langle O \rangle$.

Furthermore, from the expression for $\phi(z)$ near the boundary Eq. (A7) and Eq. (A1), the function $\dot{B}(z)$ can be calculated for small z :

$$\dot{B}(z) = zB'(z) = -\frac{1}{6}\Delta_-^2 (c - d\varepsilon^{\Delta_+ - \Delta_-})^2 z^{2\Delta_-} - \frac{1}{3}\Delta_- \Delta_+ (c - d\varepsilon^{\Delta_+ - \Delta_-}) dz^{\Delta_+ + \Delta_-} \dots \quad (\text{A9})$$

By considering homogeneous variations of the boundary condition on the metric, the one-point functions of the stress-energy tensor can be calculated using Eq. (5) as in Ref. [29].

$$\langle T^{00} \rangle = -\frac{6e^{-B(z)}}{2\kappa^2 z^4} \Big|_{z=\varepsilon}, \quad \langle T^{11} \rangle = \frac{\bar{w}}{2\kappa^2} - \langle T^{00} \rangle. \quad (\text{A10})$$

The thermal energy and pressure,

$$\epsilon = \langle T^{00} \rangle - \langle T^{00} \rangle_{T=0} \equiv [\langle T^{00} \rangle]_T, \quad p = \langle T^{11} \rangle - \langle T^{11} \rangle_{T=0} \equiv [\langle T^{11} \rangle]_T, \quad (\text{A11})$$

are finite at $\varepsilon = 0$ and equal to zero at $T = 0$. The enthalpy, $w = \epsilon + p$ is related to the constant \bar{w} :

$$w = \frac{\bar{w}}{2\kappa^2}. \quad (\text{A12})$$

After solving Eq. (A3) for B at $z = \varepsilon$ with ϕ given by Eq. (A7), we find that the energy and pressure can be expressed as

$$\epsilon = \frac{w}{4} - \frac{c[d]_T}{8\kappa^2} \Delta_- (\Delta_+ - \Delta_-), \quad p = w - \epsilon. \quad (\text{A13})$$

Therefore the temperature dependence of the expectation value of the trace anomaly is given by,

$$[\langle \theta \rangle]_T = 3p - \epsilon = \frac{c[d]_T \Delta_-}{2\kappa^2} (\Delta_+ - \Delta_-) = -\Delta_- c [\langle O \rangle]_T. \quad (\text{A14})$$

in accordance with the anomaly equation (19).

2. High temperature limit

The equations of motion can be simplified and analytically solved if one considers the high temperature limit, or the limit where $T \gg c^{1/\Delta_-}$. Since the enthalpy is related to the temperature,

$$\bar{w} = 4(\pi T)^4 (1 + \mathcal{O}(c^2 T^{-2\Delta_-})), \quad (\text{A15})$$

this limit can also be expressed as $\bar{w} \gg c^{4/\Delta_-}$. One can begin by observing that at large \bar{w} the function f varies very rapidly according to Eq. (A6). This means one can neglect variation of the function B between the boundary $z = \varepsilon$ and the horizon $z = z_H$, since z_H becomes small (as $\bar{w}^{-1/4}$). Since on the boundary $B = 0$ (up to terms of order $\varepsilon^{2\Delta_-}$, negligible here, according to Eq. (A3)), we find from Eq. (A6)

$$f(z) = 1 - \bar{w} z^4/4. \quad (\text{A16})$$

Another consequence is that ϕ , which is small at $z = \varepsilon$, remains small up to z_H ($\phi \sim c z_H^{\Delta_-} \sim c/T^{\Delta_-} \ll 1$), and the linearized approximation to Eq. (A4) is valid not only near the boundary, but all the way to the horizon. With $B = 0$ and f from Eq. (A16) we obtain

$$\left(1 - \frac{1}{4} \bar{w} z^4\right) \phi'' - \left(\frac{3}{z} + \frac{\bar{w} z^3}{4}\right) \phi' - \frac{m^2}{z^2} \phi = 0. \quad (\text{A17})$$

Equation (A17) can be solved analytically

$$\begin{aligned} \phi(z) = & c z^{\Delta_-} {}_2F_1\left(\Delta_-/4, \Delta_-/4, \Delta_-/2, \bar{w} z^4/4\right) \\ & + d z^{\Delta_+} {}_2F_1\left(\Delta_+/4, \Delta_+/4, \Delta_+/2, \bar{w} z^4/4\right), \end{aligned} \quad (\text{A18})$$

where the coefficients follow the notations of Eq. (A7) (up to terms $\mathcal{O}(\varepsilon^{\Delta_+ - \Delta_-})$, here negligible). Both linearly independent solutions are logarithmically divergent at the horizon $z = z_H$, where $\bar{w} z_H^4/4 = 1$. The condition $|\phi(z_H)| < \infty$ requires us to select the linear combination in which these divergences cancel. This fixes d in terms of c :

$$d = -c \bar{w}^{(\Delta_+ - \Delta_-)/4} D(\Delta_-), \quad (\text{A19})$$

where the function $D(\Delta_-) = 1/D(\Delta_+)$ is given by

$$D(\Delta_-) = \frac{\pi 2^{\Delta_-}}{2 - \Delta_-} \cot(\pi \Delta_-/4) \frac{\Gamma(\Delta_-/2)^2}{\Gamma(\Delta_-/4)^4}. \quad (\text{A20})$$

This solution for $\phi(z)$ can be used in Eq. (A1) to calculate $B'(z)$,

$$\begin{aligned} B'(z) = & -\frac{1}{6} \left(c \Delta_- z^{\Delta_-+1} {}_2F_1\left(\Delta_-/4, 1 + \Delta_-/4, \Delta_-/2, \bar{w} z^4/4\right) \right. \\ & \left. + d \Delta_+ z^{\Delta_++1} {}_2F_1\left(\Delta_+/4, 1 + \Delta_+/4, \Delta_+/2, \bar{w} z^4/4\right) \right)^2 \end{aligned} \quad (\text{A21})$$

From this expression, the function $B(z)$ can be calculated to order c^2 , which is the leading order in the high-temperature limit. By iteratively solving the equations of motion, the higher order corrections to B as well as to f and ϕ can be determined if necessary.

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